Thermal Instability of a Viscosity Stratified Fluid Layer Heated from Below

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THE stability of a fluid layer, under Boussinesq's approximation with viscosity $\mu(z)$ heated from below between two parallel planes (at $z=0$ and 1) is investigated under linear theory by the normal mode technique. The theoretical foundations for the correct interpretation of the problem of the onset of thermal instability in horizontal layers of fluid heated from below were laid by Rayleigh[1] in a fundamental paper. He proved that the principle of the exchange of stabilities is valid for this problem for the case of two nondeformable free boundaries. The proof for the general case is due to Pellew and Southwell.[2] They also proved the existence of a variational principle. Many authors,[3-6] after that, have discussed the stability of a fluid layer heated from below.

All the authors have considered the viscosity to be constant. For the nonheatconducting, density, and viscosity stratified fluid, Chandrasekhar,[7] Hide,[8] and Fan[9] have made an analytical treatment of its stability. Banerjee and Kalthia[10] obtained some bounds for the growth rate of arbitrary small disturbances, and a sufficient condition for the stability. Chandra[11] has modified their results, by relaxing the condition on the sign of $D^2\mu$.

In this Note, the viscosity of the fluid is taken to be stratified in the vertical direction. It is proved that the transition from stability to instability must occur via the stationary state, and the solution of the characteristic value problem at the marginal state can be expressed in terms of variational principles. Critical Rayleigh number is obtained from the first-order solution to the eigenvalue problem.

Consider a steady thermally stratified viscous fluid layer between two horizontal boundaries at $z=0$ and d, which are maintained at constant temperatures T_0 and T_1, respectively. Following the usual procedure of linearization of perturbation equations, assuming the functional dependence of the perturbed quantities on time and space coordinates of the form

$$f'(x,y,z,t)=f(z)\exp[i(k_x x+k_y y)+nt]$$

and eliminating several variables, we get the final perturbation equations

$$D((D\mu)(D^2+k^2)w+\mu(D^2-k^2)DW-nDw)=2k^2(D\mu)$$
$$\times Dw+k^2(\mu(D^2-k^2)-n)w+(g\alpha p_0 d^2/\mu_0)k^2\theta \tag{1}$$

$$(D^2-k^2-n\text{p})\theta=-(\beta d^2/\kappa)w \tag{2}$$

in dimensionless form where p is the Prandtl number. The boundary conditions are

$$\theta=w=Dw=0 \tag{3}$$

for $z=0$ and 1 when the boundaries are rigid, and

$$\theta=w=D^2w=0 \tag{4}$$

for $z=0$, and 1 when the boundaries are nondeformable free.

Multiplying Eq. (1) by w^* (the complex conjugate of w), integrating over the range of z, we get [making use of the boundary conditions and Eq. (2)]

$$\int_0^1 \mu|D^2w+k^2w|^2dz+4k^2\int_0^1\mu|Dw|^2dz+n\int_0^1(|Dw|^2$$
$$+k^2|w|^2)dz-Rk^2\int_0^1(\text{p}n^*|\theta|^2+|D\theta|^2+k^2|\theta|^2)dz=0 \tag{5}$$

where R is the Rayleigh number. Separating the real and imaginary parts of Eq. (5) we observe that if the fluid layer is heated from below ($\beta>0$), then the oscillatory modes ($n_i\neq0$) cannot exist whether n_r is zero or not; therefore, the marginal state is stationary and is characterized by $n=0$.

Since we are primarily interested in the marginal state, we put $n=0$ in Eq. (5). Thus we get

$$R=\frac{\int_0^1\mu(D^2w+k^2w)^2dz+4k^2\int_0^1\mu(Dw)^2dz}{k^2\int_0^1((D\theta)^2+k^2(\theta)^2)dz} \tag{6}$$

From Eq. (6) we observe that there exists a variational principle which can easily be proved, and so if $\delta R=0$ then $L(D^2-k^2)\theta=-Rk^2\theta$, and vice versa.

Now, for two nondeformable free boundaries, we shall obtain an approximate solution for the characteristic value problem given by

$$L(D^2-k^2)\theta=-Rk^2\theta \tag{7}$$

and the boundary conditions

$$\theta=(D^2-k^2)\theta=D^2(D^2-k^2)\theta \tag{8}$$

for $z=0$ and 1, where L is given by

$$Lw=D[(D\mu)(D^2-k^2)w+\mu(D^2-k^2)Dw]$$
$$-2k^2(D\mu)Dw-k^2\mu(D^2-k^2)w \tag{9}$$

For simplicity, we assume that $\mu=1+\delta z$ (δ is real). To obtain the solution of the characteristic value problem we express θ in terms of a complete set of functions satisfying the boundary conditions. Thus, we put

$$\theta=\sum_{n=1}^{\infty}A_n\sin n\pi z \tag{10}$$

Substituting this expression in Eq. (7) (putting $\mu=1+\delta z$), multiplying the resultant expression by $\sin n\pi z$, and integrating over the range of z, we obtain a system of linear equations for the constants A_n; and the requirement that these constants not all be zero leads to the secular relation

$$\left\|(m^2\pi^2+k^2)^3\frac{1}{2})\ \delta_{mn}+\delta Y_{mn}-\frac{1}{2}Rk^2\delta_{mn}\right\|=0 \tag{11}$$

where

$$Y_{mn}=\tfrac{1}{4}(m^2\pi^2+k^2)^3$$

if

$$m=n \qquad y_{mn}=0 \tag{12}$$

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if $m \neq n$ and $m+n$ is even,

$$Y_{mn} = \frac{4mn(m^2\pi^2+k^2)^3}{n^2-m^2}\left\{\frac{1}{(m^2\pi^2+k^2)} - \frac{1}{\pi^2(n^2-m^2)}\right\}$$

if $m+n$ is odd.

A first approximation to the solution of Eq. (11) is obtained by setting the (1,1) element of the matrix equal to zero. Thus, we get

$$R = (1+\delta/2)(\pi^2+k^2)^3/k^2 \tag{13}$$

It can be easily shown that the (1,1) element of the Eq. (11) gives lowest value of R.

The minimum value of R is given by

$$R_c = (1+\delta/2)\times 657.5 \tag{14}$$

$$k^2 = \pi^2/2 \tag{15}$$

From here we observe that if δ is positive, i.e., the viscosity of the fluid is increasing with z, then the critical Rayleigh number is increased. On the other hand, if the viscosity is decreasing with z, then the critical Rayleigh number is decreased and hence is destabilizing. Similarly if we take $\mu = e^{\delta z}$, then to the first-order approximation, R is given by

$$R = \frac{4(e^\delta - 1)(\pi^2+k^2)\{(k^2+\pi^2)^2\pi^2 + 2\delta^2k^2\pi^2\}}{k^2\delta(4\pi^2+\delta^2)} \tag{16}$$

and the critical value of R is increased or decreased according to whether the viscosity is increasing or decreasing with z. In case $\mu = \cosh(\delta z)$, then to the first-order approximation, R is given by

$$R = \frac{\{(\pi^2+k^2)^2(4\pi^2\sinh\delta+\delta^2)+4k^2\pi^2(\delta^2\sinh\delta-\delta^2)\}(\pi^2+k^2)}{k^2\delta(4\pi^2+\delta^2)} \tag{17}$$

and we note that the minimum value of the critical Rayleigh number exists for $\delta = 0.05$.

In case of rigid boundaries, for simplicity, we assume that $\mu = 1+\delta z$ and $|\delta| < < 1$ so that we may neglect the second and higher-order terms in δ, then the perturbation Eqs. (1) and (2) reduce to

$$(D^2-k^2)(1+\delta z)(D^2-k^2)w = \theta \tag{18}$$

$$(D^2-k^2)\theta = -Rk^2w \tag{19}$$

In this case the boundary conditions are

$$w = Dw = \theta = 0 \tag{20}$$

at $z=0$ and 1.

The governing Eqs. (18) and (19) with boundary conditions (20) constitute the characteristic value problem, and it can be solved by expressing the variables in terms of an appropriate complete set of functions. We expand θ in a sine series of the form (since θ is required to vanish at $z=0$ and 1)

$$\theta = \sum_{m=1}^{\infty} C_m \sin m\pi z \tag{21}$$

Having chosen θ in this manner, we next solve the equation

$$(D^2-k^2)(1+\delta z)(D^2-k^2)w = \sum_{m=1}^{\infty} C_m \sin m\pi z \tag{22}$$

obtained by inserting Eq. (21) in Eq. (18), and arrange that the solution satisfy the four remaining boundary conditions on w. The general solution of Eq. (22) can be written in the form

$$w = \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2+k^2)^2}\Big\{A_1^{(m)}\cosh kz + B_1^{(m)}\sinh kz + A_2^{(m)}\Big(z\sinh kz - \frac{\delta}{2}z^2\sinh kz + \frac{\delta}{2}\frac{z}{k}\cosh kz\Big) + B_2^{(m)}\Big(z\cosh kz - \frac{\delta}{2}z^2\cosh kz + \frac{\delta}{2}\frac{z}{k}\sinh kz\Big) + \Big(\sin m\pi z - \delta z\sin m\pi z - \frac{2\delta m\pi\cos m\pi z}{(m^2\pi^2+k^2)}\Big)\Big\} \tag{23}$$

where the constants of integration $A_1^{(m)}$, $A_2^{(m)}$, $B_1^{(m)}$, and $B_2^{(m)}$ are determined by the boundary conditions: $w = Dw = 0$ at $z=0$ and 1. These latter conditions lead to four equations which on solving give

$$A_1^{(m)} = 2\delta m\pi D \tag{24}$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\Big[k - (-1)^{m+1}\sinh k - \delta D\Big(2[(-1)^{m+1}\sinh k + k(-1)^{m+1}\cosh k + \sinh k\cosh k + k] + k - \frac{3}{2}(-1)^{m+1}\sinh k\Big)\Big] \tag{25}$$

$$A_2^{(m)} = \frac{m\pi}{\Delta}\Big[(\sinh k\cosh k - k) - (-1)^{m+1}(k\cosh k - \sinh k) + \delta\Big(\frac{k}{2} - \frac{3}{2}(-1)^{m+1}\sinh k - \sinh k\cosh k + \frac{1}{2k}\sinh^2 k + \frac{3}{2}k(-1)^{m+1}\cosh k - 2kD\sinh^2 k - 2k^2D(-1)^{m+1}\sinh k\Big)\Big] \tag{26}$$

$$B_2^{(m)} = -\frac{m\pi}{\Delta}\Big[\sinh^2 k - (-1)^{m+1}k\sinh k + \delta\Big(\frac{1}{2} - \cosh^2 k + \frac{1}{2k}\sinh k\cosh k + \frac{3}{2}(-1)^{m+1}k\sinh k - \frac{1}{2}(-1)^{m+1}\cosh k + \frac{1}{2k}(-1)^{m+1}\sinh k - 2k^2D - 2D((-1)^{m+1}k\sinh k + (-1)^{m+1}k^2\cosh k + k\cosh k\sinh k)\Big)\Big] \tag{27}$$

where

$$\Delta = (1-\delta)(\sinh^2 k - k^2), \quad D = (m^2\pi^2+k^2)^{-1} \tag{28}$$

Now substituting for θ and w from Eqs. (21) and (23) in Eq. (19), multiplying the resultant expression by $\sin n\pi z$, and integrating over z, we obtain a system of linear homogeneous equations for the constants $C_m/(m^2\pi^2+k^2)$; and the requirement that these constants not all be zero leads to the secular equation in simplified form

$$\Big\| -\frac{\delta_{mn}}{2Rk^2D_n^3} + 2\delta mn\pi(1-(-1)^{m+n})D_nD_m + \Big\{ -2kn\pi D_n^2(1+(-1)^{n+1}\cosh k) + \frac{\delta}{2}\Big[2n\pi D_n^2(-1)^{n+1}\sinh k + 4kn\pi D_n^2 \times (-1)^{n+1}\cosh k - 8k^2n\pi^2D_n^3(-1)^{n+1}\sinh k \Big] - \frac{\delta}{k}(-1)^{n+1}n\pi D_n^2 k\sinh k \Big\} A_2^{(m)} + \Big\{ -2kn\pi D_n^2(-1)^{n+1}\sinh 2k + \frac{\delta}{2}\Big(4kn\pi D_n^2(-1)^{n+1}\sinh k + 2n\pi D_n^2[1+(-1)^{n+1}\cosh k] - 8k^2n\pi D_n^3[1+(-1)^{n+1}\cosh k] \Big) - \delta n\pi D_n^2[1+(-1)^{n+1}\cosh k] \Big\} B_2^{(m)} + \frac{1}{2}\delta_{mn} - \delta X_{nm} \Big\| = 0 \qquad (29)$$

where

$$X_{nm} = 0$$

if $m+n$ is even and $m \neq n$,

$$X_{nm} = \frac{1}{4} \qquad (30)$$

if $m = n$,

$$X_{nm} = \frac{4mn}{(n^2-m^2)}\left[\frac{1}{m^2\pi^2+k^2} - \frac{1}{\pi^2(n^2-m^2)}\right]$$

if $m+n$ is odd, and

$$D_j = (j^2\pi^2+k^2)^{-1} \qquad (31)$$

A first approximation to the solution of Eq. (29) is obtained by setting the (1,1) element of the matrix equal to zero. Thus, we find after making use of the expressions for $A_2^{(1)}$, $B_2^{(1)}$

$$R = (1+\delta/2)(\pi^2+k^2)^3/k^2\left(1 - \frac{16k\pi^2\cosh^2 k/2}{(\sinh k + k)(\pi^2+k^2)^2}\right) \qquad (32)$$

We observe that, apart from the factor $(1+\delta/2)$, this expression for R is identical to what was obtained by Pellew and Southwell[2] for the simple Bénard problem by the variational method in the first approximation for the case of two rigid boundaries. Consequently,

$$R_c = (1+\delta/2) \times 1715, \qquad k_{\min} = 3.12$$

Here, also, we observe that if δ is positive, i.e., the viscosity is increasing upwards, the critical Rayleigh number is increased, and if δ is negative R_c is decreased, similar to the case of free boundaries when $\mu = 1 + \delta z$.

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Computation of Flow Past a Rotating Cylinder with an Energy-Dissipation Model of Turbulence

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Nomenclature

$C_\mu, C_1, C_2, C_3, C_c$	= constants in turbulence model
eff	= effective value
k	= turbulent kinetic energy
ℓ	= length scale of turbulent eddy
p	= static pressure
r	= radial distance from the axis of symmetry
R	= radius of cylinder
R_t	= turbulent Reynolds number, $k^2/\nu\epsilon$
Re_∞	= freestream Reynolds number, $U_\infty R/\nu$
Ri	= swirl Richardson number, defined by Eqs. (2) and (11)
U	= velocity in the x direction
U_∞	= freestream velocity
V_θ	= circumferential velocity
W	= velocity directed normal to the surface
x	= coordinate measured along the surface
z	= coordinate measured normal to the surface

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